AXIALLY SYMMETRIC FLOWS OF IDEAL AND VISCOUS FLUIDS FILLING THE WHOLE SPACE

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We show that the Cauchy problem for the Navier-Stokes and Euler's equations has, on the whole, a unique solution in the case of axisymmetric flows of an incompressible fluid. We also show that, when viscosity disappears, then the solutions of Navier-Stokes equations tend to the solutions of Euler's equations. In the general three-dimensional case, the unique solvability has only been proved for some isolated cases (for the viscous flows in [1 and 2] and for the nonviscous flows in [3 and 4]). Complete solutions exist however in the plane case for both problems ([5 and 6] for viscous and [7 and 8] for nonviscous flows).

1. Statement of the problem and a priori estimates. Let us consider the Cauchy problem for the Navier-Stokes equations (problem A) and for the Euler's equations (problem B) in the case of an incompressible fluid filling the whole space R^3

$$\frac{\partial \mathbf{v}}{\partial t} - \nu\Delta \mathbf{v} + (\mathbf{v}, \nabla)\mathbf{v} = -\nabla P + \mathbf{F}(x, t), \quad \operatorname{div} \mathbf{v} = 0, \quad \mathbf{v}(x, 0) = \mathbf{a}(x) \quad (1.1)$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}, \nabla)\mathbf{v} = -\nabla P + \mathbf{F}(x, t), \quad \operatorname{div} \mathbf{v} = 0, \quad \mathbf{v}(x, 0) = \mathbf{a}(x) \quad (1.2)$$

where $\mathbf{v} = \mathbf{v}(x, t)$ and $P = P(x, t)$ are the velocity and pressure respectively, $x = (x_1, x_2, x_3)$ denotes a point belonging to R^3; $t \in [0, T]$ (where $0 < T$ is any number); $\mathbf{F}(x, t)$ and $\mathbf{a}(x)$ are known vectors solenoidal in R^3 and ν is a positive constant.

We shall call the vector $\mathbf{v} = (v_r, v_\theta, v_z)$ axisymmetric if $v_\theta = 0$ with v_r and v_z are independent of θ and we shall call a function axisymmetric, if it is independent of θ (r, θ and z are cylindrical coordinates).

We shall now introduce the following functional spaces:

$H_0(R^3)$ and $H_1(R^3)$ are the complements of the set of all vectors, smooth, finite and solenoidal in R^3 over the norms of the following scalar products:

$$(\mathbf{u}_1, \mathbf{u}_2)_{H_0} = \int_{R^3} \mathbf{u}_1\mathbf{u}_2 dx, \quad (\mathbf{u}_1, \mathbf{u}_2)_{H_1} = \int_{R^3} \operatorname{rot} \mathbf{u}_1 \operatorname{rot} \mathbf{u}_2 dx$$

$H_0'(Q_T)$ and $H_1'(Q_T)$ are the complements of the set of all vectors smooth in $Q_T = R^3\times[0, T]$, finite and solenoidal in R^3 over the norm of the following scalar products:

$$(\mathbf{u}_1, \mathbf{u}_2)_{H_0'} = \int_0^T (\mathbf{u}_1, \mathbf{u}_2)_{H_0} dt, \qquad (\mathbf{u}_1, \mathbf{u}_2)_{H_1'} = \int_0^T (\mathbf{u}_1, \mathbf{u}_2)_{H_1} dt$$

$H_2(R^3)$ and $H_3(R^3)$ are the complements of the set of all vectors smooth and axisymmetric in R^3 and finite in the semi-plane passing through the x_3-axis, over the following norms:

$$\|\mathbf{u}\|_{H_1}=\|\mathbf{u}\|_{H_0}+\|\mathbf{u}\|_{H_1}+\left\|\frac{\operatorname{rot}\mathbf{u}}{r}\right\|_{L_2}+\left\|\frac{\operatorname{rot}\mathbf{u}}{r}\right\|_{M},\quad \|\mathbf{u}\|_{H_2}=\|\mathbf{u}\|_{H_1}+\|\operatorname{rot}\mathbf{u}\|_{M}$$

$H_2'(Q_T)$ and $H_3'(Q_T)$ are the complements of the set of all vectors smooth in Q_T axisymmetric and solenoidal in R^3 and finite on the semi-plane passing through the x_3-axis, over the norms

$$\|\mathbf{u}\|_{H_1'}=\int_0^T\|\mathbf{u}\|_{H_2}dt,\qquad \|\mathbf{u}\|_{H_2'}=\int_0^T\|\mathbf{u}\|_{H_3}dt$$

Definition 1.1. The vector $\mathbf{v}(x, t)\in H_1'(Q_T)$ shall be called the generalized solution of the problem (1.1) in the cylinder Q_T. This vector will have finite

$$\max\|\mathbf{v}(x, t)\|_{H_0(R^3)},\qquad \max\|\mathbf{v}(x, t)\|_{L_p(R^3)}\quad (0\leqslant t\leqslant T)$$

for some p $(3<p\leqslant 6)$ and will satisfy the following integral identity $(\nabla\phi\equiv(\nabla\phi_1, \nabla\phi_2, \nabla\phi_3))$

$$-\int_{R^3}\mathbf{a}(x)\varphi(x,0)\,dx+\int_0^T\int_{R^3}[-\mathbf{v}\varphi_t+\nu\nabla\mathbf{v}\nabla\varphi+(\mathbf{v},\nabla)\mathbf{v}\varphi-\mathbf{F}\varphi]\,dxdt=0 \quad (1.3)$$

for any vector $\phi(x, t)$ smooth in Q_T, finite and solenoidal in R^3, for which $\phi(x, T)=0$.

Definition 1.2. The vector $\mathbf{v}(x, t)\in H_1'(Q_T)$ shall be called the generalized solution of the problem (1.2) in the cylinder Q_T. This vector will have finite

$$\max\|\mathbf{v}(x, t)\|_{H_0(R^3)},\qquad \max\|\operatorname{rot}\mathbf{v}(x, t)\|_{M(R^3)}\qquad (0\leqslant t\leqslant T)$$

and will satisfy the following integral identity

$$-\int_{R^3}\mathbf{a}(x)\varphi(x,0)\,dx+\int_0^T\int_{R^3}[-\mathbf{v}\varphi_t+(\mathbf{v},\nabla)\mathbf{v}\varphi-\mathbf{F}\varphi]\,dx\,dt=0 \qquad (1.4)$$

for any vector $\phi(x, t)$ smooth in Q_T and finite and solenoidal in R^3, for which $\phi(x, T)=0$.

Vectors $\mathbf{a}(x)$ and $\mathbf{F}(x, t)$ are assumed such, that the corresponding integrals in (1.3) and (1.4) have a meaning, e.g. they can be generalized vector functions concentrated on some surfaces or curves.

As usual, we verify that the classical solutions of (1.1) and (1.2) are generalized solutions in the sense of the above definitions and conversely, that the generalized solutions with all continuous derivatives appearing in (1.1) and (1.2) are classical solutions of these problems.

We shall now give a priori estimates for solutions of the problems A and B, which are supposed to be smooth, axisymmetric and decaying sufficiently rapidly at infinity. As we know, the energy equation yields (axial symmetry need not be assumed here).

Lemma 1.1. The following estimate holds for solutions of the problems A and B

$$\|\mathbf{v}(x, t)\|_{L_2(R^3)}\leqslant\|\mathbf{a}\|_{L_2(R^3)}+\int_0^t\|\mathbf{F}(x, \tau)\|_{L_2(R^3)}d\tau \qquad (1.5)$$

Vorticity equations of the problems A and B

$$\omega_t-\nu\Delta\omega+(\mathbf{v},\nabla)\omega-(\omega,\nabla)\mathbf{v}=\mathbf{f} \qquad (1.6)$$

$$\omega_t+(\mathbf{v},\nabla)\omega-(\omega,\nabla)\mathbf{v}=\mathbf{f}\qquad (\omega=\operatorname{rot}\mathbf{v},\ \mathbf{f}=\operatorname{rot}\mathbf{F}) \qquad (1.7)$$

will yield further estimates.

In the axisymmetric case, (1.6) and (1.7) can be written as

$$\frac{\partial\omega}{\partial t}-\nu\left(\frac{\partial^2\omega}{\partial r^2}+\frac{\partial^2\omega}{\partial z^2}+\frac{1}{r}\frac{\partial\omega}{\partial r}-\frac{\omega}{r^2}\right)+v_r\frac{\partial\omega}{\partial r}+v_z\frac{\partial\omega}{\partial z}-\frac{v_r}{r}\omega=f \qquad (1.8)$$

$$\frac{\partial\omega}{\partial t}+v_r\frac{\partial\omega}{\partial r}+v_z\frac{\partial\omega}{\partial z}-\frac{v_r}{r}\omega=f \qquad (1.9)$$

and by virtue of axial symmetry we have

$$\omega_r = \omega_z = 0, \ \omega_\theta = \omega(r, z, t); \ f_r = f_z = 0, \ f_\theta = f(r, z, t)$$

Continuity equation in this case has the form

$$\frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} + \frac{v_r}{r} = 0 \tag{1.10}$$

L e m m a 1.2. The following estimates hold for the axisymmetric solution of the problem A(B) ($1 < p$):

$$\left\| \frac{\omega(x,t)}{r} \right\|_{L_p(R^3)} \leqslant \left\| \frac{\operatorname{rot} \mathbf{a}}{r} \right\|_{L_p(R^3)} + \int_0^t \left\| \frac{\operatorname{rot} \mathbf{F}(x,\tau)}{r} \right\|_{L_p(R^3)} d\tau \tag{1.11}$$

$$\left\| \frac{\omega(x,t)}{r} \right\|_{M(R^3)} \leqslant \left\| \frac{\operatorname{rot} \mathbf{a}}{r} \right\|_{M(R^3)} + \int_0^t \left\| \frac{\operatorname{rot} \mathbf{F}(x,\tau)}{r} \right\|_{M(R^3)} d\tau \tag{1.12}$$

$$\nu \int_0^t \int_{R^3} \left(\nabla \left| \frac{\omega}{r} \right|^{p/2} \right)^2 dx\, d\tau \leqslant \frac{p}{4(p-1)} \left\| \frac{\operatorname{rot} \mathbf{a}}{r} \right\|_{L_p(R^3)}^p + \frac{p^2}{4(p-1)} \int_0^t \left\| \frac{\operatorname{rot} \mathbf{F}(x,\tau)}{r} \right\|_{L_p(R^3)} \times$$

$$\times \left(\left\| \frac{\operatorname{rot} \mathbf{a}}{r} \right\|_{L_p(R^3)} + \int_0^\tau \left\| \frac{\operatorname{rot} \mathbf{F}(x,\eta)}{r} \right\|_{L_p(R^3)} d\eta \right)^{p-1} d\tau \tag{1.13}$$

P r o o f. Multiplying (1.8) by

$$\left| \frac{\omega}{r} \right|^{p-1} \operatorname{sign} \frac{\omega}{r}$$

and integrating over the semi-plane E passing through the x_3-axis, we obtain (1.14)

$$\frac{1}{p} \frac{d}{dt} \int_E r \left| \frac{\omega}{r} \right|^p dr\, dz + \nu \frac{4(p-1)}{p^2} \int_E r \left(\nabla \left| \frac{\omega}{r} \right|^{1/2 p} \right)^2 dr\, dz = \int_E f \left| \frac{\omega}{r} \right|^{p-1} \operatorname{sign} \frac{\omega}{r}\, dr\, dz$$

Applying now the Hölder inequality to the right-hand side of (1.14) we find, that

$$Z_p^{p-1} \frac{d}{dt} Z_p \leqslant \left\| \frac{f}{r} \right\|_{Lp(E)} Z_p^{p-1}, \qquad Z_p(t) \equiv \int_E r \left| \frac{\omega}{r} \right|^p dr\, dz \tag{1.15}$$

which integrated over t yields (1.11). Putting $p \to \infty$ in (1.11) we obtain (1.12), while the inequality (1.13) follows directly from (1.14) and (1.11). Lemma 1.2 is proved.

L e m m a 1.3. The following estimate holds for the axisymmetric solution of the problem A(B):

$$\| \omega(x,t) \|_{L_2(R^3)} \leqslant \| \operatorname{rot} \mathbf{a} \|_{L_2(R^3)} + \int_0^t \| \operatorname{rot} \mathbf{F}(x,\tau) \|_{L_2(R^3)} d\tau + \int_0^t B(\tau)\, d\tau \tag{1.16}$$

$$B(t) = \left(\| \mathbf{a} \|_{L_2(R^3)} + \int_0^t \| \mathbf{F}(x,\tau) \|_{L_2(R^3)} d\tau \right) \left(\left\| \frac{\operatorname{rot} \mathbf{a}}{r} \right\|_{M(R^3)} + \int_0^t \left\| \frac{\operatorname{rot} \mathbf{F}(x,\tau)}{r} \right\|_{M(R^3)} d\tau \right) \tag{1.17}$$

P r o o f. Multiplying (1.8) by $r\omega$ and integrating, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_E r\omega^2 dr\, dz + \nu \int_E r \left[(\nabla \omega)^2 + \left(\frac{\omega}{r} \right)^2 \right] dr\, dz = \int_E v_r \omega^2\, dr\, dz + \int_E r f \omega\, dr\, dz$$

$$\frac{1}{2} \frac{d}{dt} \int_E r\omega^2 dr\, dz \leqslant \int_E \frac{\omega}{r} r v_r \omega\, dr\, dz + \int_E (r^{1/2} f)(r^{1/2} \omega)\, dr\, dz \leqslant$$

$$\leqslant\left[\max_{x\in R^3}\left|\frac{\omega}{r}\right|\left(\int_E r v_r^2\,dr\,dz\right)^{1/2}+\left(\int_E r f^2\,dr\,dz\right)^{1/2}\right]\left(\int_E r\omega^2\,dr\,dz\right)^{1/2} \tag{1.18}$$

Integration of (1.18) with respect to t from 0 to t, with (1.5) and (1.13) taken into account, yields (1.16) and (1.17), which proves Lemma 1.3.

L e m m a 1.4. The following estimate holds for the axisymmetric solution of the problem A(B) (C is an absolute constant):

$$\|\omega(x,t)\|_{L_4(R^3)}\leqslant\|\operatorname{rot}\mathbf{a}\|_{L_4(R^3)}+\int_0^t\|\operatorname{rot}\mathbf{F}(x,\tau)\|_{L_4(R^3)}\,d\tau+\int_0^t B_1(\tau)\,d\tau \tag{1.19}$$

$$B_1(t)=C\left(\left\|\frac{\operatorname{rot}\mathbf{a}}{r}\right\|_{M(R^3)}+\int_0^t\left\|\frac{\operatorname{rot}\mathbf{F}(x,\tau)}{r}\right\|_{M(R^3)}d\tau\right)\left[\left(\|\mathbf{a}\|_{L_2(R^3)}+\int_0^t\|\mathbf{F}(x,\tau)\|_{L_2(R^3)}\,d\tau\right)\times\right.$$

$$\left.\times\left(\|\operatorname{rot}\mathbf{a}\|_{L_2(R^3)}+\int_0^t\|\operatorname{rot}\mathbf{F}(x,\tau)\|_{L_2(R^3)}\,d\tau+\int_0^t B(\tau)\,d\tau\right)^3\right]^{1/4} \tag{1.20}$$

Its proof follows the lines of that of Lemma 1.3. Here (1.8) is multiplied by $r\omega^3$ and we utilize the previous estimates together with the imbedding estimate $H_0(R^3)\cap H_1(R^3)$ in $L_4(R^3)$ (see e.g. [9]).

L e m m a 1.5. For any function $\phi(x)$ we have

$$\|\varphi\|_{L_p(R^3)}\leqslant\frac{s}{p}\|\varphi\|_{L_s(R^3)}+\frac{p-s}{p}\|\varphi\|_{M(R^3)}\ (2\leqslant s\leqslant p<\infty) \tag{1.21}$$

P r o o f. The inequality (1.21) follows from the even more obvious inequality

$$\|\varphi\|_p^p\leqslant\|\varphi\|_M^{\alpha}\|\varphi\|_{p-\alpha}^{p-\alpha}$$

by putting $\alpha_p{}'=p-s$ and applying the Young's inequality

$$ab\leqslant\frac{a^p}{p}+\frac{b^{p'}}{p'}\qquad\left(a,\ b\geqslant 0,\ p>1,\ p'=\frac{p}{p-1}\right)$$

L e m m a 1.6. The estimate $(1<p)$:

$$\|D_x\mathbf{u}\|_{L_p(R^3)}\leqslant Cp\|\operatorname{rot}\mathbf{u}\|_{L_p(R^3)} \tag{1.22}$$

holds for any vector $\mathbf{u}(x)$ solenoidal in R^3 and sufficiently rapidly diminishing at infinity. Here C is a constant independent of $\mathbf{u}$ and p.

P r o o f. We know [10] that a vector $\mathbf{u}$ solenoidal in R^3, satisfies

$$\mathbf{u}=\operatorname{rot}\mathbf{B},\qquad\Delta\mathbf{B}=-\operatorname{rot}\mathbf{u},\qquad\mathbf{B}_{\infty}=0$$

from which (1.22) can be obtained using the estimate of the solution of the Dirichlet problem for the Poisson's equation, obtained in [11].

L e m m a 1.7. The following estimates ($2\leqslant p$, C_1 and C_2 are absolute constants) hold for the axisymmetric solution of the problem A(B):

$$\|\mathbf{v}(x,t)\|_{M(R^3)}\leqslant C_1\left(\|\mathbf{a}\|_{L_2(R^3)}+\int_0^t\|\mathbf{F}(x,\tau)\|_{L_2(R^3)}\,d\tau+\|\operatorname{rot}\mathbf{a}\|_{L_4(R^3)}+\right.$$

$$\left.+\int_0^t\|\operatorname{rot}\mathbf{F}(x,\tau)\|_{L_4(R^3)}\,d\tau+\int_0^t B_1(\tau)\,d\tau\right) \tag{1.23}$$

$$\|\mathbf{v}(x,t)\|_{L_p(R^3)}\leqslant C_2\left(\|\mathbf{a}\|_{L_2(R^3)}+\int_0^t\|\mathbf{F}(x,\tau)\|_{L_2(R^3)}\,d\tau+\|\operatorname{rot}\mathbf{a}\|_{L_4(R^3)}+\right.$$

$$\left.+\int_0^t\|\operatorname{rot}\mathbf{F}(x,\tau)\|_{L_4(R^3)}\,d\tau+\int_0^t B_1(\tau)\,d\tau\right) \tag{1.24}$$

Proof. Estimate (1.23) follows by imbedding [12] from (1.5) and (1.19) on the basis of Lemma 1.6, while the estimate (1.24) follows from (1.5) and (1.23) by Lemma 1.5.

Lemma 1.8. The following estimates hold for the axisymmetric solution of the problem A(B)

$$\|\omega(x,t)\|_{L_p(R^3)} \leqslant C_1\left(\|\operatorname{rot}\mathbf{a}\|_{L_p(R^3)} + \int_0^t \|\operatorname{rot}\mathbf{F}(x,\tau)\|_{L_p(R^3)}d\tau + \int_0^t B_2(\tau)\,d\tau\right) \quad (1.25)$$

$$\|\omega(x,t)\|_{M(R^3)} \leqslant C_2\left(\|\operatorname{rot}\mathbf{a}\|_{M(R^3)} + \int_0^t \|\operatorname{rot}\mathbf{F}(x,\tau)\|_{M(R^3)}d\tau + \int_0^t B_2(\tau)\,d\tau\right) \quad (1.26)$$

$$B_2(t) = \left(\left\|\frac{\operatorname{rot}\mathbf{a}}{r}\right\|_{M(R^3)} + \int_0^t \left\|\frac{\operatorname{rot}\mathbf{F}(x,\tau)}{r}\right\|_{M(R^3)}d\tau\right)\left(\|\mathbf{a}\|_{L_2(R^3)} + \int_0^t \|\mathbf{F}(x,\tau)\|_{L_2(R^3)}d\tau + \right.$$
$$\left. + \|\operatorname{rot}\mathbf{a}\|_{L_4(R^3)} + \int_0^t \|\operatorname{rot}\mathbf{F}(x,\tau)\|_{L_4(R^3)}d\tau + \int_0^t B_1(\tau)\,d\tau\right) \quad (1.27)$$

where $2 \leqslant p$, while C_1 and C_2 are absolute constants.

The proof of (1.25) follows that of Lemma 1.3, where (1.8) is multiplied by $r|\omega|^{p-1}\operatorname{sign}\omega$ and the estimate (1.24) is used. Estimate (1.26) follows from (1.25) in the limit as $p \to \infty$ and this completes the proof.

2. Uniqueness theorems. Lemma 2.1. The generalized solution of the problem A(B) satisfies, at any $t_1 \in [0, T]$, the following integral identity

$$\int_{R^3} \mathbf{v}(x,t_1)\varphi(x,t_1)\,dx - \int_{R^3} \mathbf{a}(x)\varphi(x,0)\,dx + \int_0^{t_1}\int_{R^3}[-\mathbf{v}\varphi_t + \nu\nabla\mathbf{v}\nabla\varphi +$$
$$+ (\mathbf{v},\nabla)\mathbf{v}\varphi - \mathbf{F}\varphi]\,dx\,d\tau = 0 \quad (2.1)$$

for any $\phi(x, t)$ given in the definition of the generalized solution of the problem A(B). The proof follows that for the case of plane parallel flows of an ideal fluid [8].

Theorem 2.1. There exist not more than one generalized solution of the problem A.

Proof. Let $\mathbf{v}(x, t)$ and $\mathbf{v}_1(x, t)$ be two solutions of the problem A and let $\mathbf{u} = \mathbf{v}_1 - \mathbf{v}$. Let us write the identity (2.1) for $\mathbf{v}$ and $\mathbf{v}_1$ and subtract them from each other, putting $\phi = A_n^2\mathbf{u}$, the operator A_n is defined by

$$A_n\mathbf{u} = \int_{Q_{t_1}} \omega_{h_n}(x-y,\, t-\tau)\,\mathbf{u}(y,\tau)\,dy\,d\tau$$

where $\omega_{h_n}(x-y, t-\tau)$ is the averaging kernel [13], $Q_{t_1} = R^3\times[0, t_1]$ and $h_n \to 0$. In the limit as $n \to \infty$, we obtain

$$\frac{1}{2}\|\mathbf{u}(x,t)\|^2_{H_0(R^3)} + \nu\int_0^{t_1}\|\mathbf{u}(x,\tau)\|^2_{H_1(R^3)}d\tau = \int_0^{t_1}\int_{R^3}(\mathbf{u}\times\operatorname{rot}\mathbf{u})\,\mathbf{v}\,dx\,d\tau \quad (2.2)$$

Passing to the limit is valid here, since $A_n^2\mathbf{u}$ converges to $\mathbf{u}$ in $L_{6,2}(Q_{t_1})$ (see [8] for the definition of $L_{p,r}$ spaces) and $\nabla A_n^2\mathbf{u}$ converges to $\nabla\mathbf{u}$ in $L_2(Q_{t_1})$, while $A_n^2\mathbf{u}(x, t)$ and $A_n\mathbf{u}(x, t)$ converges to $\mathbf{u}(x, t)$ in $L_2(R^3)$.

Applying the Hölder inequality to the right-hand side of (2.2) and taking into account the imbedding estimates we obtain ($2 < p < 6$, $p^{-1} + q^{-1} = ½$ and C is an absolute constant):

$$\frac{1}{2}\|\mathbf{u}\|^2_{H_0} + \nu\int_0^{t_1}\|\mathbf{u}\|^2_{H_1}d\tau \leqslant C\max_{0\leqslant t\leqslant T}\|\mathbf{v}\|_q\int_0^{t_1}\|\mathbf{u}\|_{H_1}^{\frac{5p-6}{2p}}\|\mathbf{u}\|_{H_0}^{\frac{6-p}{2p}}d\tau \quad (2.3)$$

Next we apply the Young's inequality with the indices

$$r=\frac{4p}{5p-6},\qquad s=\frac{4p}{6-p}$$

to the integral expression in the right-hand side of (2.3), to obtain

$$\frac{1}{2}\|\mathbf{u}\|^2_{H_0}+\nu\int_0^{t_1}\|\mathbf{u}\|^2_{H_1}d\tau\leqslant C_1\varepsilon^r\int_0^{t_1}\|\mathbf{u}\|^2_{H_1}d\tau+C_2\varepsilon^{-s}\int_0^{t_1}\|\mathbf{u}\|^2_{H_0}d\tau \tag{2.4}$$

where the constants C_1 and C_2 are independent of u, ε and t_1. Choosing now ε such that $C_1\ \varepsilon^r<\nu$, we obtain from (2.4) (the constant C being independent of $\mathbf{u}$ and t)

$$\|\mathbf{u}\|^2_{H_0}\leqslant C\int_0^{t_1}\|\mathbf{u}\|^2_{H_0}d\tau$$

This implies that $\mathbf{u}\equiv 0$, which proves Theorem 2.

T h e o r e m 2.2. There exist not more than one generalized solution of the problem B.

P r o o f. We follow the procedure used in the proof of Theorem 2.1. Putting $Z(t)=\|\mathbf{u}\|=\|\mathbf{u}\|_{H_0(R^3)}$, we obtain

$$Z\frac{dZ}{dt}=-\int_{R^3}(\mathbf{u},\nabla)\,\mathbf{v}\mathbf{u}\,dx \tag{2.5}$$

From the definition 1.2, Lemmas 1.5 and 1.6 and the inclusion theorem [12] it follows that the vector $\mathbf{u}$ is bounded in Q_T: $|\mathbf{u}(x,t)|\leqslant M$. Now applying to (2.5) the Hölder inequality and Lemmas 1.5 and 1.6 we obtain (C is an absolute constant)

$$Z\frac{dZ}{dt}\leqslant CM^{\varepsilon}\int_{R^3}|\mathbf{u}|^{2-\varepsilon}\,|\nabla\mathbf{v}|\,dx\leqslant CM^{\varepsilon}\|\nabla\mathbf{v}\|_{2/\varepsilon}Z^{2-\varepsilon}\leqslant$$

$$\leqslant CM^{\varepsilon}\frac{2}{\varepsilon}\|\operatorname{rot}\mathbf{v}\|_{2/\varepsilon}Z^{2-\varepsilon}\leqslant CM^{\varepsilon}\frac{2}{\varepsilon}(\|\operatorname{rot}\mathbf{v}\|_2+\|\operatorname{rot}\mathbf{v}\|_M)Z^{2-\varepsilon}$$

from which, taking into account the fact that

$$\|\operatorname{rot}\mathbf{v}\|_2+\|\operatorname{rot}\mathbf{v}\|_M\leqslant M_1 \text{ when } t\in[0,T]$$

we obtain, on integrating with respect to t,

$$Z(t)\leqslant M\,(2CM_1t)^{1/\varepsilon} \tag{2.6}$$

Putting $\varepsilon\to 0$ in (2.6) we find, that $Z(t)=0$ when $t\in[0,\tau_0]$ and $\tau_0=(4CM_1)^{-1}$. Repeating this procedure for the segments $[\tau_0,2\tau_0]$, $[2\tau_0,3\tau_0]$ etc., we can show that $z(t)\equiv\equiv 0$ on $[0,T]$ which implies that $\mathbf{u}(x,t)\equiv 0$ in Q_T which proves the theorem.

3. Existence of a solution to the problem A. Here we prove the following theorem:

T h e o r e m 3.1. Let $\mathbf{a}(x)\in H_2(R^3)$ and $\mathbf{F}(x,t)\in H_2{}'(Q_T)$. Then an axisymmetric generalized solution of the problem A exists.

Let $\{\mathbf{b}^{(n)}(x)\}$ and $\{\mathbf{F}^{(n)}(x,t)\}$ be the sequences of vectors, infinitely differentiable, axisymmetric and solenoidal in R^3, and finite in the semi-plane passing through the x_3-axis which converge to $\mathbf{a}(x)$ and $\mathbf{F}(x,t)$ respectively in $H_2(R^3)$ and $H_2{}'(Q_T)$; let further $\{D^{(n)}\}$ be a set of concentric spheres which, together, fill the whole R^3-space and such, that the vectors $\mathbf{b}^{(n)}$ and $\mathbf{F}^{(n)}$ are equal to zero outside $D^{(n)}$. We shall define the vector $\mathbf{u}^{(n)}(x,t)$ in $Q_T{}^{(n)}=D^{(n)}\times[0,T]$ as the axisymmetric solution of the problem

$$\mathbf{u}_t^{(n)}-\nu\Delta\mathbf{u}^{(n)}+(\mathbf{u}^{(n)},\nabla)\,\mathbf{u}^{(n)}=-\nabla P^{(n)}+\mathbf{F}^{(n)},\qquad \operatorname{div}\mathbf{u}^{(n)}=0 \tag{3.1}$$

$$\mathbf{u}^{(n)}(x,0)=\mathbf{b}^{(n)};\qquad \mathbf{u}^{(n)}\cdot\mathbf{n}\,|_{S(n)}=0,\qquad \operatorname{rot}\mathbf{u}^{(n)}|_{S(n)}=0 \tag{3.2}$$

where $S^{(n)}$ is the boundary of the sphere $D^{(n)}$ and $\mathbf{n}$ denotes the unit outward vector normal to $S^{(n)}$.

We shall call "the generalized solution of the problem (3.1) and (3.2) in $Q_T^{(n)}$ ", the axisymmetric vector $\mathbf{u}^{(n)}(x, t)$ satisfying the conditions of Definition 1.1 (if it is continued as its null value outside $D^{(n)}$), together with the following integral identity

$$-\int_{D^{(n)}} \mathbf{b}^{(n)}(x)\varphi^{(n)}(x,0)\,dx + \int_0^T\int_{D^{(n)}} [-\mathbf{u}^{(n)}\varphi_t^{(n)} + \nu\nabla\mathbf{u}^{(n)}\nabla\varphi^{(n)} + \\ + (\mathbf{u}^{(n)}, \nabla)\mathbf{u}^{(n)}\varphi^{(n)} - \mathbf{F}^{(n)}\varphi^{(n)}]\,dx\,dt = 0 \qquad (3.3)$$

for any vector $\phi^{(n)}(x, t)$ smooth in $Q_T^{(n)}$ and solenoidal and axisymmetric in $D^{(n)}$, for which $\phi^{(n)}(x, T) = 0$ and for which conditions (3.2.2) and (3.2.3) hold.

We shall use the Galerkin method to construct a generalized solution of the problem (3.1) and (3.2). We shall fix the value of n and, for the time being drop the superscript in $D^{(n)}$, $S^{(n)}$, $Q_T^{(n)}$, $\mathbf{b}^{(n)}$, $\mathbf{F}^{(n)}$, $\mathbf{u}^{(n)}$ and $\phi^{(n)}$. Let now Ω be the region of intersection of the sphere D with the semi-plane passing through the x_3-axis, let Σ be the boundary of Ω and let $\{\phi_k(r, z)\}$ be a sequence, normed in $L_2(\Omega)$, of the eigenfunctions of the problem

$$-(\varphi_{rr} + \varphi_{zz}) = \lambda\varphi, \qquad \varphi/_\Sigma = 0 \qquad (3.4)$$

orthogonal and complete, as we know, in the spaces $L_2(\Omega)$ and $\overset{\circ}{W}{}_2^{(1)}(\Omega)$, and let $\omega_k(r, z) = r^2\phi_k(r, z)$; the sequence $\{\omega_k\}$ is obviously orthonormal and complete in the $L_{2, r^{-4}}(\Omega)$ space of functions quadratically summable in Ω with the weight r^{-4}.

Let us now obtain the vector $\mathbf{u}_k(x)$ in D as a solution of the problem

$$\operatorname{div}\mathbf{u}_k = 0, \quad \operatorname{rot}\mathbf{u}_k = \boldsymbol{\omega}_k, \quad \mathbf{u}_k\cdot\mathbf{n}\,|_S = 0 \quad (\boldsymbol{\omega}_k = (0, \omega_k(r, z), 0)) \quad (3.5)$$

Since $\operatorname{div}\omega_k = 0$, the problem (3.5) has a unique solution [10]. Let $\sigma(r, z) = (\operatorname{rot}\mathbf{b})_\theta$. Since $\sigma(r, z) \in L_{2, r^{-4}}$, it can be expanded into a Fourier series in $\omega_k(r, z)$

$$\sigma(r, z) = \sum_{i=1}^{\infty} \alpha_i\omega_i(r, z)$$

We shall take the vector

$$\mathbf{u}^{(m)}(x, t) = \sum_{i=1}^{m} A_i^{(m)}(t)\,\mathbf{u}_i(x) \qquad (3.6)$$

as an m-th Galerkin approximation, with the following conditions (3.7)

$$\int_\Omega r\left\{\left(\frac{\partial\omega^{(m)}}{\partial t} + u_r^{(m)}\frac{\partial\omega^{(m)}}{\partial r} + u_z^{(m)}\frac{\partial\omega^{(m)}}{\partial z} - u_r^{(m)}\frac{\omega^{(m)}}{r}\right)\frac{\omega_k}{r^2} + \nu\left[\frac{\partial\omega^{(m)}}{\partial r}\frac{\partial}{\partial r}\left(\frac{\omega_k}{r^2}\right) + \right.\right. \\ \left.\left. + \frac{\partial\omega^{(m)}}{\partial z}\frac{\partial}{\partial z}\left(\frac{\omega_k}{r^2}\right) - \frac{1}{r}\frac{\partial\omega^{(m)}}{\partial r}\frac{\omega_k}{r^2} + \frac{\omega^{(m)}}{r}\frac{\partial}{\partial r}\left(\frac{\omega_k}{r^2}\right) + \frac{\omega^{(m)}}{r^2}\frac{\omega_k}{r^2}\right] - f\frac{\omega_k}{r^2}\right\} dr\,dz = 0$$

$$A_k^{(m)}(0) = \alpha_k \qquad \left(\omega^{(m)}(r, z, t) = \sum_{i=1}^{m} A_i^{(m)}(t)\,\omega_i(r, z)\right) \qquad (3.8)$$

imposed on the functions $A_k(t)$ $(k = 1, 2, \ldots, m)$.

Lemma 3.1. Vectors $\mathbf{u}^{(m)}(x, t)$ $(m = 1, 2, \ldots)$ are uniquely definable by the relations (3.6) to (3.8), are axisymmetric and solenoidal in D, have, together with $\mathbf{u}_t^{(m)}$ second derivatives in x which are continuous in Q_T, and satisfy the conditions (3.2).

Proof. We insert (3.6) into (3.7) to find $A_k^{(m)}(t)$. This yields a first order system of ordinary differential equations, which has a unique solution satisfying the initial conditions (3.8). The remaining assertions emerge directly from the properties of the vectors $\mathbf{u}_k(x)$.

Lemma 3.2. From the sequence $\{\omega^{(m)}\}$ a subsequence (for which we retain the previous notation) can be formed, which weakly converges in $L_2(\Omega_T)$ $(\Omega_T = \Omega\times[0, T])$ to a function $\omega(r, z, t)$ so, that $\partial\omega^{(m)}/\partial r$, $\partial\omega^{(m)}/\partial z$ and $\omega^{(m)}/r$ converge weakly in $L_2(\Omega_T)$ to

$\partial\omega/\partial r$, $\partial\omega/\partial z$ and ω/r respectively.

P r o o f. Multiplying each of Eqs. of (3.7) by the corresponding $A_k^{(m)}$ and adding them together after the transformations analogous to those applied in Lemma 1.2, we obtain the following estimate uniform in m and $t \in [0, T]$

$$\int_0^t\int_\Omega \left[\left(\frac{\partial\omega^{(m)}}{\partial r}\right)^2 + \left(\frac{\partial\omega^{(m)}}{\partial z}\right)^2 + \left(\frac{\omega^{(m)}}{r}\right)^2\right] dr\, dz d\tau \leqslant C$$

It remains to apply the theorems on weak compactness of a sphere in a Hilbert space and on weak closure of the generalized differentiation.

L e m m a 3.3. From the sequence $\{\omega^{(m)}\}$ a subsequence (for which we retain the previous notation) can be formed, which converges to $\omega(r, z, t)$ weakly in $L_2(\Omega)$ and uniformly in $t \in [0, T]$.

P r o o f. Proceeding from the relations (3.7) we can easily show that the following continuous functions

$$l_{m,k}(t) = \int_\Omega \omega^{(m)} \varphi_k r\, dr dz$$

form, at any fixed value of k and $m \geqslant k$, an uniformly bounded and uniformly continuous family. Let us now select, out of each family, a subsequence converging uniformly on $[0, T]$.

Diagonalization then yields a subsequence converging uniformly for each fixed k when $m \to \infty$.

Taking now into account the completeness of the system $\{\phi_k\}$ in $L_2(\Omega)$ we can easily show that the corresponding subsequence of sequences $\{\omega^{(m)}\}$ converges weakly in $L_2(\Omega)$ uniformly in $t \in [0, T]$.

L e m m a 3.4. From the sequence $\{\mathbf{u}^{(m)}(x, t)\}$ a subsequence can be formed, which converges weakly in $W_2^{(1)}(D)$ uniformly in $t \in [0, T]$ and strongly in $L_p(\Omega_T)$ for $1 < p < 4$ to a vector $\mathbf{u}(x, t)$ which is axisymmetric, solenoidal and has the vorticity given by $\omega(x, t) = (0, \omega, 0)$.

Its proof follows from (1.22) and the continuity of the imbedding operator.

L e m m a 3.5. Vector $\mathbf{u}(x, t)$ represents a generalized solution of the problem (3.1), (3.2).

P r o o f. By Lemma 3.4 it remains to show that $\mathbf{u}(x, t)$ satisfies the identity (3.3). Let $\psi(r, z, t)$ be an arbitrary function continuous in Ω_T together with its first derivatives and equal to zero on Σ and when $t = T$. If now λ_k is the eigenvalue of the problem (3.4) corresponding to the eigenfunction ϕ_k, then the functions $\psi_k(r, z) = (1 + \lambda_k)^{-1/2} \phi_k(r, z)$ form a complete system orthonormal in $\overset{\circ}{W}{}_2^{(1)}(\Omega)$, and $\psi(r, z, t)$ has the following expansion

$$\psi(r, z, t) = \sum_{k=1}^{\infty} c_k(t)\, \psi_k(r, z) \tag{3.9}$$

converging in $W_2^{(1)}(\Omega)$ at any $t \in [0, T]$. It can easily be shown that convergence will be uniform on $[0, T]$ and that the series for $\partial\psi/\partial t$ also converges uniformly in $t \in [0, T]$, in $W_2^{(1)}(\Omega)$.

From this, the imbedding theorem implies the uniform on $[0, T]$ convergence of both series in any $L_p(\Omega)$ $(1 < p < \infty)$ and even more, the convergence in any $L_p(\Omega_T)$. Denoting the m-th partial sum of the series (3.9) by $\psi^{(m)}$, multiplying each of the Eqs. of (3.7) by $(1 + \lambda_k)^{-1/2} c_k$, summing over k from 1 to m and integrating with respect to t, we obtain

$$-\int_\Omega r\omega^{(m)}(r, z, 0)\, \psi^{(m)}(r, z, 0)\, dr\, dz + \int_0^T\int_\Omega r\left[-\omega^{(m)} \frac{\partial\psi^{(m)}}{\partial t} + \left(u_r^{(m)} \frac{\partial\omega^{(m)}}{\partial r} + \right.\right.$$

$$\left. + u_z^{(m)} \frac{\partial\omega^{(m)}}{\partial z} - u_r^{(m)} \frac{\omega^{(m)}}{r}\right) \psi^{(m)} + \nu\left(\frac{\partial\omega^{(m)}}{\partial r}\frac{\partial\psi^{(m)}}{\partial r} + \frac{\partial\omega^{(m)}}{\partial z}\frac{\partial\psi^{(m)}}{\partial z} - \right.$$

$$+\frac{1}{r}\frac{\partial\omega^{(m)}}{\partial r}\psi^{(m)}+\frac{\omega^{(m)}}{r}\frac{\partial\psi^{(m)}}{\partial r}+\frac{\omega^{(m)}}{r^2}\psi^{(m)}\Big)-f\psi^{(m)}\Big]\,dr\,dz\,dt=0 \qquad (3.10)$$

Taking into account the character of convergence of $\psi^{(m)}$, $u_r^{(m)}$, $u_z^{(m)}$ and $\omega^{(m)}$, we can easily show that the limiting process as $m\to\infty$ is applicable to (3.10) (this corresponds to the formal deletion of the index m). Let now $\phi(x, t)$ be an arbitrary vector described in the definition of the generalized solution of the problem (3.1) and (3.2), the function $\psi(r, z, t)$ can be obtained as a solution of the following problem:

$$\frac{1}{r}\frac{\partial(r\psi)}{\partial r}=\varphi_z,\quad -\frac{\partial\psi}{\partial z}=\varphi_r,\qquad \psi(0, 0, t)=0 \qquad (3.11)$$

It can easily be shown that the problem (3.11) has a unique solution and, that this solution $\psi(r, z, t)$ is equal to zero on Σ and when $t = T$. Putting $\psi = (0, \psi, 0)$ and assuming that by (3.11) rot $\psi = \phi$, we can transform the identity (3.10) into

$$-\int_D \mathbf{b}(x)\varphi(x, 0)\,dx+\int_0^T\int_D[-\mathbf{u}\varphi_t+\nu\nabla\mathbf{u}\,\nabla\varphi+(\mathbf{u}, \nabla)\mathbf{u}\varphi-\mathbf{F}\varphi]\,dx\,dt=0$$

which proves the Lemma.

Lemma 3.6. Vorticity of the generalized solution of (3.1) and (3.2) represented by the vector $\omega(x, t)$, has second order derivatives with respect to x and first order derivatives with respect to t, summable to degree p $(1 < p < 2)$ over the cylinder Q_T.

Proof. Proceeding from identity (3.3) we can easily show that $\omega(x, t)$ coincides almost everywhere in Q_T with $\omega'(x, t)$ which is the solution of

$$\omega_t'-\nu\Delta\omega'=\mathbf{g}(x, t),\qquad \omega'|_S=0,\qquad \omega'|_{t=0}=\sigma(x)$$
$$\mathbf{g}\equiv(\omega, \nabla)\mathbf{u}-(\mathbf{u}, \nabla)\omega+f,\quad \sigma(x)=\omega(x, 0)$$

Lemmas 3.2 and 3.4 imply that $\mathbf{g}(x, t)\in L_p(Q_T)$ for $1 < p < 2$, and the proof now follows from the results of Solonnikov [14].

Lemma 3.7. Generalized solution of the problem (3.1), (3.2) represented by the vector $\mathbf{u}(x, t)$, has derivatives with respect to x and t, of any order and continuous in Q_T.

The proof can be obtained by repeated application of Lemma 3.6).

Thus we have proved the existence of the (classical) solution of the problem (3.1), (3.2) for any value of n.

To complete the proof of Theorem 3.1 we shall now consider the sequence $\{\mathbf{u}^{(n)}(x, t)\}$.

Assuming that the sequences $\{\mathbf{b}^{(n)}\}$ and $\{\mathbf{F}^{(n)}\}$ are uniformly bounded in $H_2(R^3)$ and $H_2'(Q_T)$ respectively and applying Lemmas 1.1 to 1.3, we obtain the following estimate:

$$\|\mathbf{u}\|^{(n)}(x, t)\|_{H_2(D^{(n)})}\leqslant C \qquad (3.12)$$

uniform in n and $t\in[0, T]$.

From (3.12) it follows that from $\{\mathbf{u}^{(n)}\}$ a subsequence can be formed, which converges weakly in $H_0(R^3)$ and $H_1(R^3)$ uniformly in $t\in[0, T]$ to some vector $\mathbf{v}(x, t)$. Taking (3.3) to the limit as $n\to\infty$, we obtain (1.3) which completes the proof.

Theorem 3.2. If $\mathbf{a}(x)\in H_3(R^3)$ and $\mathbf{F}(x, t)\in H_3'(Q_T)$, then the axisymmetric generalized solution $\mathbf{v}(x, t)$ of the problem A has a bounded vorticity

$$\max\|\,\mathrm{rot}\ \mathbf{v}(x, t)\|_{M(R^3)}\qquad (0\leqslant t\leqslant T)$$

The proof follows that of Theorem 3.1. We assume that $\{\mathbf{b}^{(n)}\}$ and $\{\mathbf{F}^{(n)}\}$ converge to $\mathbf{a}$ and $\mathbf{F}$ in $H_3(R^3)$ and $H_3'(Q_T)$ respectively. Using Lemmas 1.1 to 1.3 and 1.8, we obtain the estimate

$$\|\mathbf{u}^{(n)}(x, t)\|_{H_3(D^{(n)})}\leqslant C \qquad (3.13)$$

uniform in n and $t\in[0, T]$.

Using (3.13) together with Lemma 1.5, we can form from the elements of $\{\mathbf{u}^{(n)}\}$, for any $p\geqslant 2$, a subsequence converging weakly in $H_0(R^3)$ and $H_1(R^3)$ uniformly in $t\in[0, T]$. Vorticities of the elements of this subsequence converge weakly in $L_p(R^3)$ uniformly in $t\in[0, T]$. Applying now the theorem on weak closure of generalized differentiation, we obtain the estimate $\|\text{rot}\,\mathbf{v}(x, t)\|_{L_p(R^3)} \leqslant C$ uniform in $p \geqslant 2$ and $t \in [0, T]$, from which the proof follows.

4. Existence of a generalized solution to the problem B.

Theorem 4.1. Let $\mathbf{a}(x) \in H_3(R^3)$ and $\mathbf{F}(x, t) \in H_3'(Q_T)$. Then an axisymmetric solution to the problem B exists.

Proof. Let $\{\nu^{(n)}\}$ be a sequence of positive numbers converging to zero and let $\mathbf{v}^{(n)}(x, t)$ be the axisymmetric generalized solution of the problem A, where the value of viscosity is given by $\nu = \nu^{(n)}$. Taking into account the fact that the estimates given in Section 1 are independent of viscosity ν, and Theorem 3.2, we obtain the following estimates uniform in n and $t \in [0, T]$:

$$\|\mathbf{v}^{(n)}(x, t)\|_{H_0(R^3)} \leqslant C, \quad \|\mathbf{v}^{(n)}(x, t)\|_{H_1(R^3)} \leqslant C, \quad \|\text{rot}\,\mathbf{v}^{(n)}(x, t)\|_{M(R^3)} \leqslant C$$

from which it follows, that a subsequence can be formed from $\{\mathbf{v}^{(n)}(x, t)\}$ converging weakly in $H_0(R^3)$ and $H_1(R^3)$ uniformly in $t \in [0, T]$ to some vector $\mathbf{v}(x, t)$, whose vorticity has an upper bound $\max\|\text{rot}\,\mathbf{v}(x, t)\|_{(MR^3)}$, $(0 \leqslant t \leqslant T)$.

On passing to the limit as $n \to \infty$, the identity

$$-\int_{R^3} \mathbf{a}(x)\,\varphi(x, 0)\,dx + \int_0^T\int_{R^3} [-\mathbf{v}^{(n)}\varphi_t + (\mathbf{v}^{(n)}, \nabla)\,\mathbf{v}^{(n)}\varphi - \mathbf{F}\varphi]\,dx\,dt + $$
$$+\,\mathbf{v}^{(n)}\int_0^T\int_{R^3} \text{rot}\,\mathbf{v}^{(n)}\,\text{rot}\,\varphi\,dx\,dt = 0$$

yields the relation (1.4), which proves the Theorem.

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ON THE KINEMATICS, NONEQUILIBRIUM THERMODYNAMICS, AND RHEOLOGICAL RELATIONSHIPS IN THE NONLINEAR THEORY OF VISCOELASTICITY

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Within the scope of the customary thermodynamics of irreversible processes (TIP) (a linear connection between thermodynamic fluxes and forces, symmetry of the kinetic coefficients), and utilizing the relationship derived herein between reversible, irreversible, and total strain rates, a system of governing equations is constructed for the simplest viscoelastic media in the presence of arbitrary finite reversible deformations.

These equations are investigated in the case of sufficiently small reversible deformations; a "second-order" theory is constructed taking into account the physical as well as the geometrical, nonlinearity in the system. It is hence taken into account that the kinetic coefficients will be tensor functions of the tensor of reversible deformations. This latter leads to "deformation anisotropy" of the heat conduction and diffusion. Expressions are written down for entropy production in the system for the simplest model media.

The "second-order" theory is extended to the case of isothermal deformation of viscoelastic media with many relaxation times. The solution of a number of problems for the simplest flows (simple shear, tension) of viscoelastic media showed a good enough qualitative agreement between the constructed theory and experiment. Also questions about the inversion of the Jaumann tensor derivative ("Jaumann integration") are considered.

A large quantity of papers (see the survey [1]) is devoted to a theoretical description of viscoelastic media. In the phenomenological construction of a theory of viscoelasticity, as in the construction of continuum models generally [2 and 3], invariance considerations, the geometry of finite deformations, and thermodynamics are utilized, while the thermodynamics of irreversible processes (TIP) is used for dissipative media. Biot [4 and 5] made a sufficiently complete investigation of linear viscoelasticity under conditions of small velocities of this kind.

Let us refer to the work of Kluitenberg in which the thermodynamic derivation of governing equations for various media is expounded [6 to 9].

Among the earliest investigations on the nonlinear theory of viscoelasticity is the paper [10]; however, the kinematics of viscoelastic phenomena remained unclarified in this work, and there is a total absence of a thermodynamic analysis of the phenomena.

The development of a theory of nonlinear behavior of dissipative media is often connected with the extension of TIP [11]. In opposition to such a viewpoint, an attempt is made herein to utilize the customary version of TIP with linear phenomenological laws and Onsager reciprocity relationships, to derive the governing equations of a nonlinear viscoelastic medium with physical and geometric nonlinearities.

We shall often rely on [2 and 12] without detailed referral in expounding the theory of de-